$\exists\, w_1 \in H_0^1$ s.t. $\|w_1\|_{L^2} = 1$

$$\lambda_1 = \inf_{\substack{u \in C_0^\infty \\ \|u\| \neq 0}} E(u) = E(w_1)$$

where $E(u) = \dfrac{\int |\nabla u|^2 - V u^2}{\int u^2}$.

prop) $\lambda_1 = \inf_{\substack{u \in H_0^1, \\ \|u\| \neq 0}} E(u)$

pf) H_0^1 is the closure of C_0^∞ in H^1.

$$\Rightarrow \inf_{\substack{u \in H_0^1 \\ \|u\|_{L^2} \neq 0}} E(u) = \inf_{\substack{u \in C_0^\infty \\ \|u\| \neq 0}} E(u)$$

prop) Given $v \in H_0^1$, $\exists\, \varepsilon > 0$ s.t.

$I(s) = E(w_1 + s v) \geq E(w_1) = \lambda_1$

for $|s| < \varepsilon$. .

Thm) $\int_\Omega -\nabla w_1 \cdot \nabla \varphi + V w_1 \varphi + \lambda_1 w_1 \varphi = 0$ — (*)

$\forall \varphi \in H_0^1$.

pf) $I(s) = E(w_1 + s\varphi)$

$$\frac{I'(0)}{2 I(0)} = \frac{1}{2} \frac{d}{ds} \log I(s) \Big|_{s=0}$$

$\Rightarrow$ (*)!

Def) We call $u \in H_0^1$ a weak solution to $\Delta u + (V + \lambda_1) u = 0$ of zero Dirichlet condition if $\int_\Omega -\nabla u \cdot \nabla \varphi + V u \varphi + \lambda_1 u \varphi = 0$

$\forall \varphi \in H_0^1$.

Hence w_1 is a weak sol to the elliptic eq.

$$\Delta w_1 + (V+\lambda_1) w_1 = 0$$

w/ zero Dirichlet condition.

By Theorem 8.29 in the textbook (Sandro Salsa)

(see also [Evans], [GT]. weak solutions)

$$w_1 \in H^m(\Omega) \quad \forall m \geq 2$$

(since $V \in C^\infty$, $\partial\Omega$ is C^∞)

where $H^m(\Omega)$ (or $W^{2,m}$) consists of functions w/ i-th order weak derivatives ($i \leq m$) belonging to $L^2(\Omega)$.

$W^{1,p}(\Omega)$ consist of functions u w/ weak derivatives $\partial_i u$.

s.t. $u, \partial_i u \in L^p$.

$$\|u\|_{W^{1,p}} = \left(\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p \right)^{1/p}$$

In particular, $H^1 = W^{1,2}$

Similarly, $\|u\|_{W^{k,p}} = \left(\sum_{i=0}^{k} \|\nabla^i u\|_{L^p}^p \right)^{1/p}$

$$W^{k,2} = H^k.$$

$W_0^{k,p}$ is the closure of C_0^∞ in $W^{k,p}$.

ex) $H_0^1 = W_0^{1,2}$. $H_0^m = W_0^{m,2}$.

Sobolev embedding) (ch 7. Thm [GT2])

If $p<n$, then $W^{1,p} \subset L^{\frac{np}{n-p}}$.

i.e. if $u \in W^{1,p}$ then $u \in L^{\frac{np}{n-p}}$.

If $p>n$, then given any $u \in W^{1,p}$ there exists $u^* \in C^\alpha$ (where $\alpha = 1-\frac{n}{p}$) such that the set $\{u \neq u^*\}$ has measure zero.

Then. $w_1 \in H_0^m$

$\Rightarrow \exists\ w_1^* \in C^{k,\alpha}$ s.t

$\{w_1 \neq w_1^*\}$ has measure zero.

where $k<m$. $m-k \leq C(n)$.

$\therefore \exists\ w_1^* \in C_0^\infty$ s.t $\Delta w_1^* + c(u+\lambda)w_1 = 0$.

we may replace w_1 with w_1^*.

Then.. we can show the existence of $w_2 \in H_0^1$ s.t. $\|w_2\|_{L^2}=1$.

$$\lambda_2 = \inf_{u \in X_1} E(u) = E(w_2)$$

Then. w_2 is a weak sol to $\Delta w_2 + (V+\lambda_2) w_2 = 0$ of zero Dirichlet condition.

By the theorem 8.29 in Salsa.

$w_2 \in H_0^m \quad \forall m \geq 2.$

$\Rightarrow \exists w_2^* \in C_0^\infty$ s.t. $\{w_2 \neq w_2^*\}$ has m.0.

$\Delta w_2^* + (V+\lambda_2) w_2 = 0.$

$\Rightarrow$ Iterate !!

proof of weak convergence
of bounded sequence in H^1
introduced in lecture 10.

(Cf. Riesz representation theorem)

Let $\|u_k\|_{H^1} \leq 1$. $\forall k \in \mathbb{N}$.

Claim: $B = \text{span}\{u_1, u_2, \cdots\}$
has countable othornormal basis.

pf of claim) Gram-Schmidt!!

Let u_{k_1} be the 1^{st} non-zero term.

Define $w_1 = u_{k_1} / \|u_{k_1}\|_{H^1}$

Let u_{k_2} satisfy that $u_{k_2} \notin \text{span}\{w_1\}$
$u_j \in \text{span}\{w_1\}$ $\forall j < k_2$.

Define $w_2 = \dfrac{u_{k_2} - \langle u_{k_2}, w_1 \rangle_{H^1} w_1}{\| u_{k_2} - \langle u_{k_2}, w_1 \rangle_{H^1} w_1 \|_{H^1}}$

Let u_{k_3} satisfy $u_{k_3} \notin \operatorname{span}\{w_1, w_2\}$

$u_j \in \operatorname{span}\{w_1, w_2\}$

$$w_3 \hat{=} \frac{u_{k_3} - \langle u_{k_3}, w_1 \rangle w_1 - \langle u_{k_3}, w_2 \rangle w_2}{\| \quad \text{''} \quad \|_{H^1}}$$

We iterate to obtain $\{w_\ell\}_{\ell=1}^{\infty}$.

Then. $\|w_\ell\|_{H^1} = 1$. $\langle w_\ell, w_m \rangle_{H^1} = 0$, $\ell \neq m$

$\operatorname{span}\{u_1, \dots, u_\ell\} \subset \operatorname{span}\{w_1, \dots, w_\ell\}$

$\Rightarrow \operatorname{span}\{u_1, \dots\} \subset \operatorname{span}\{w_1, \dots\}$

i.e. $\{w_\ell\}_{\ell=1}^{\infty}$ is the desired basis.

Next, we denote

$$\langle u_k, w_\ell \rangle = a_k^\ell .$$

$$\Rightarrow u_k = \sum_{\ell=1}^{\infty} a_k^\ell w_\ell .$$

$$1 \geq \|u_k\|_{H^1}^2 = \sum_{\ell=1}^{\infty} (a_k^\ell)^2$$

$$\Rightarrow |a_k^\ell| \leq 1 \quad \forall k, \ell .$$

a_k^1 has a convergent subseq. $a_{k_j}^1$

$$\Rightarrow \lim_{k_j \to +\infty} \langle u_{k_j}, w_1 \rangle = \lim_{k_j \to +\infty} a_{k_j}^1 = \bar{a}^1$$

for some $\bar{a}^1 \in [-1, 1]$.

$a_{k_j}^2$ has a conv. subseq $a_{k_{j_m}}^2$.

$$\Rightarrow \lim_{k_{j_m}} \langle u_{k_{j_m}}, w_2 \rangle = \bar{a}^2 \in [-1, 1]$$

By iterating this process

we have a seq $f_i : \mathbb{N} \to \mathbb{N}$.

s.t. $f_i(m) < f_i(m+1)$

$$f_{i+1}(\mathbb{N}) \subset f_i(\mathbb{N})$$

$$\lim_{m\to+\infty} \langle u_{f_i(m)}, w_i \rangle = \lim a^i_{f_i(m)} = \bar{a}^i$$

$\hat{u}_i = u_{f_i(i)}$. is a subseq of u_i.

In addition. $\lim_{i\to+\infty} \langle \hat{u}_i, w_j \rangle = \bar{a}^j \in [-1, 1]$

Claim 2) $\sum_{j=1}^{m} |\bar{a}^j|^2 \le 1 \quad \forall m \in \mathbb{N}$.

proof) $\tilde{u}_m \triangleq \sum_{j=1}^{m} \bar{a}^j w_j \in H'$

$$\Rightarrow \langle \hat{u}_i, \tilde{u}_m \rangle_{H'} = \sum_{j=1}^{m} \langle \hat{u}_i, w_j \rangle_{H'} \bar{a}^j$$

$$\to \sum_{j=1}^{m} |\bar{a}^j|^2 = \|\tilde{u}_m\|^2_{H'}$$

$$\Rightarrow \langle \hat{u}_i, \tilde{u}_m \rangle \longrightarrow \|\tilde{u}_m\|^2_{H^1}$$

as $i \to +\infty$

$$|\langle \hat{u}_i, \tilde{u}_m \rangle_{H^1}| \leq \|\hat{u}_i\|_{H^1} \|\tilde{u}_m\|_{H^1}$$

$$= \|\tilde{u}_m\|_{H^1}$$

$$\Rightarrow \|\tilde{u}_m\|^2_{H^1} \leq \|\tilde{u}_m\|_{H^1}$$

$$\therefore \|\tilde{u}_m\|_{H^1} \leq 1.$$

$$\Leftrightarrow \sum_{j=1}^{m} (\bar{a}^j)^2 \leq 1 \quad \forall m \geq 1$$

So, we have $\bar{u} = \sum_{j=1}^{\infty} \bar{a}^j w_j \in H^1$

w/ $\|\bar{u}\|_{H^1} \leq 1$. $\blacksquare$

Finally, we claim

$$\lim_{i\to+\infty} \langle \hat{u}_i, v\rangle_{H^1} = \langle \bar{u}, v\rangle_{H^1}$$

for every $v \in H^1$.

proof) given $v \in H^1$

we consider $\bar{v} = \sum_{j=1}^{\infty} \langle v, w_j\rangle w_j$

$= \mathrm{proj}_B v$.

Then, $\langle \hat{u}_i, v\rangle_{H^1} = \langle \hat{u}_i, \bar{v}\rangle_{H^1}$

$\langle \bar{u}, v\rangle_{H^1} = \langle \bar{u}, \bar{v}\rangle_{H^1}$

Given $\varepsilon > 0$, $\exists\, m \in \mathbb{N}$ s.t

$$\left\| \bar{v} - \sum_{l=1}^{m} \langle \bar{v}_l, w_l\rangle w_l \right\|_{H^1} < \varepsilon.$$

Define $\hat{v} = \sum_{l=1}^{m} \langle \bar{v}_l, w_l\rangle w_l$

$$\langle \hat{u}_i, \bar{v}\rangle_{H^1} = \langle \hat{u}_i, \bar{v}-\tilde{v}\rangle_{H^1} + \langle \hat{u}_i, \tilde{v}\rangle_H$$

$$|\langle \hat{u}_i, \bar{v}-\tilde{v}\rangle_{H^1}| \le \|\hat{u}_i\|_{H^1} \cdot \|\bar{v}-\tilde{v}\|_{H^1}$$

$$\le 1 \cdot \varepsilon = \varepsilon.$$

$$\langle \hat{u}_i, \tilde{v}\rangle = \sum_{l=1}^{m} \langle \hat{u}_i, w_l\rangle \langle v, w_l\rangle$$

$$\to \sum_{l=1}^{m} \langle \bar{u}, w_l\rangle \langle v, w_l\rangle$$

$$= \langle \bar{u}, \tilde{v}\rangle$$

as $i \to +\infty$.

$$|\langle \bar{u}, \bar{v}-\tilde{v}\rangle_{H^1}| \le \varepsilon.$$

$\Rightarrow$ for sufficiently large i,

$$|\langle \hat{u}_i, v\rangle - \langle \bar{u}, v\rangle| < 3\varepsilon.$$

$\therefore$ $\lim_{i\to+\infty} \langle \hat{u}_i, v\rangle = \langle \bar{u}, v\rangle$

for any $v \in H^1$.

Suppose that $\|u_k\|_{H^1} \leq 1$. $\forall k \in \mathbb{N}$

and $\lim_{k \to +\infty} \langle u_k, v \rangle_{H^1} = \langle u, v \rangle_{H^1}$

for all $v \in H^1_0$.

Namely, $u \in H^1$ is a weak limit of the seq. $\{u_k\}$

This does NOT imply that u_k has a limit in H^1.

Namely, we do NOT know if

$\exists u \in H^1$ s.t. $\lim_{k \to \infty} \|u_k - u\|_{H^1} = 0$.

Ex) Let $\Omega = [0,\pi] \subset \mathbb{R}$

$$u_k(x) = \sqrt{\frac{2}{\pi}}(1+k^2)^{-1/2}\sin kx$$

$$\|u_k\|_{H^1}^2 = \frac{2}{\pi}(1+k^2)^{-1}\int_0^{\pi} (\sin kx)^2 + (k\cos kx)^2\,dx$$

$$= \frac{2}{\pi}(1+k^2)^{-1}\left(\frac{\pi}{2} + k^2\frac{\pi}{2}\right) = 1.$$

Claim: 0 is a weak limit

i.e. $\lim_{k\to+\infty}\langle u_k, v\rangle_{H^1} = 0 \quad \forall v \in H_0^1(\Omega)$

pf) $H_0^1 \subset L^2$, $v = \sum_{k=1}^{\infty} a_k u_k$

where $\sum_{k=1}^{\infty} a_k^2 < +\infty$.

However, $\langle u_k, v\rangle = a_k\|u_k\| = a_k \to 0$

But $\lim_{k\to+\infty}\|u_k - 0\|_{H^1} = 1 \neq 0$

Ex) $\ell^2(\mathbb{R}^\infty) = \{ x \in \mathbb{R}^\infty \mid \|x\|_2^2 = \sum_{i=1}^{\infty} x_i^2 < +\infty \}$

$x, y \in \ell^2(\mathbb{R}^\infty) \Rightarrow \langle x, y \rangle_{\ell^2} = \sum_{i=1}^{\infty} x_i y_i$

$\ell^2(\mathbb{R}^\infty)$ is a Hilbert space

Let $u_k = e_k = (0, 0, \ldots, 0, \overset{k\text{-th}}{1}, 0, \ldots)$

Then. $\|u_k\|_{\ell^2} = 1$.

and $\lim_{k \to +\infty} \langle u_k, v \rangle_{\ell^2} = 0 \quad \forall v \in \ell^2$

However. $\lim \|u_k - 0\|_{\ell^2} = 1$.